

# Distribution of Extreme Points of the Error Curve of Best Approximation by Incomplete Polynomials

CHENGMIN YANG

*Department of Mathematics, West Virginia Institute of Technology,  
Montgomery, West Virginia 25136*

*Communicated by Edward B. Saff*

Received October 30, 1989; accepted in revised form March 8, 1993

Let  $\Pi_{n,m} = \{(x+1)^n P_m(x) : P_m(x) \in \Pi_m\}$  be the set of incomplete polynomials of type  $(n, m)$  on  $[-1, 1]$ . We consider the distribution of the extreme points of the error curve in the best approximation of  $f \in C[-1, 1]$  with  $f(-1) = 0$  by incomplete polynomials of type  $(n, m)$  on  $[-1, 1]$  and give a Kadec-type result for this setting. In particular, we improve the rate of convergence of the original Kadec result. © 1994 Academic Press, Inc.

## 1. INTRODUCTION

Let  $\Pi_{n,m} = \{(x+1)^n P_m(x) : P_m(x) \in \Pi_m\}$  be the set of incomplete polynomials of type  $(n, m)$ . Approximation of  $f \in C[-1, 1]$  with  $f(-1) = 0$  by incomplete polynomials has been studied by many authors and many results have been obtained (see [4, 5, 7–9]). Analogous to Chebyshev polynomials, Saff and Varga introduced the constrained Chebyshev polynomial  $T_{n,m}$  of  $\Pi_{n,m}$  and proved several properties of  $T_{n,m}$  in [8]. Their results are given on the interval  $[0, 1]$ . To make the presentation cleaner and to be consistent with classical results, we consider the interval  $[-1, 1]$  instead of  $[0, 1]$ .

**DEFINITION 1.1.** Let  $H_{n,m}$  minimize  $\{\|(x+1)^{n+m} - q\| : q \in \Pi_{n,m-1}\}$ . Then  $T_{n,m} = H_{n,m}/\|H_{n,m}\|$  is called the constrained Chebyshev polynomial of  $\Pi_{n,m}$ , where  $\|f\| = \max_{-1 \leq x \leq 1} |f(x)|$ .

The following two theorems were given in [8] on the interval  $[0, 1]$ .

**THEOREM 1.2.** For each  $T_{n,m}$  there exist  $m+1$  distinct points  $\{\xi_j^{(n,m)}\}_{j=0}^m$  with

$$2 \left( \frac{n}{m+n} \right)^2 - 1 \leq \xi_0^{(n,m)} < \dots < \xi_m^{(n,m)} = 1,$$

$$T_{n,m}(\xi_j^{(n,m)}) = (-1)^{m-j} = (-1)^{m-j} \|T_{n,m}\|,$$

and  $T_{n,m}$  is monotone in each of  $(\xi_j^{(n,m)}, \xi_{j+1}^{(n,m)})$ ,  $j=0, 1, \dots, m-1$ ,  $(-1, \xi_0^{(n,m)})$ ,  $(-\infty, -1)$ , and  $(1, \infty)$ .

**THEOREM 1.3.** If  $P \in \Pi_{n,m}$  and  $M = \max_{0 \leq k \leq m} \{|P(\xi_k^{(n,m)})|\}$  then

$$|P(x)| \leq M |T_{n,m}(x)| \quad \text{for all } x \notin (\xi_0^{(n,m)}, 1)$$

and

$$|P^{(k)}(x)| \leq M |T_{n,m}^{(k)}(x)| \quad \text{for all } x \notin (-1, 1), k=1, 2, \dots, n+m.$$

The above points  $\{\xi_j^{(n,m)}\}_{j=0}^m$  are called the extreme points of  $T_{n,m}$ .

In this paper we discuss the distribution of the extreme points and zeros of  $T_{n,m}$  and the distribution of the extreme points of the error curve in the best approximation by incomplete polynomials which extends the results of Kadec in [3] for approximation by ordinary polynomials with an improved rate.

## 2. MAIN RESULTS

Let  $\Pi_{n,m}$ ,  $T_{n,m}$ , and  $\xi_j^{(n,m)}$  be defined as above over the interval  $[-1, 1]$  and let  $\{\sigma_j^{(n,m)}\}_{j=1}^m$  denote the unique zero of  $T_{n,m}$  in each  $(\xi_{j-1}^{(n,m)}, \xi_j^{(n,m)})$ ,  $j=1, \dots, m$ .

The first two theorems concern the distribution of  $\{\xi_j^{(n,m)}\}$  and  $\{\sigma_j^{(n,m)}\}$ .

**THEOREM 2.1.** For the  $\xi_j^{(n,m)}$ ,  $j=0, \dots, m$ , and  $\sigma_j^{(n,m)}$ ,  $j=1, \dots, m$ , defined as above,

$$\begin{aligned} 2 \left( \frac{n}{n+m} \right)^2 - 1 &\leq \xi_0^{(n,m)} < \xi_0^{(n,m-1)} < \xi_1^{(n,m)} < \dots < \xi_{m-1}^{(n,m)} < \xi_{m-1}^{(n,m-1)} \\ &= \xi_m^{(n,m)} = 1 \end{aligned} \quad (2.1)$$

and

$$\sigma_1^{(n,m)} < \sigma_1^{(n,m-1)} < \sigma_2^{(n,m)} < \dots < \sigma_{m-1}^{(n,m-1)} < \sigma_m^{(n,m)} < 1. \quad (2.2)$$

*Proof.* The first inequality in (2.1) is from Theorem 1.2. Now, since  $T_{n,m-1} \in \Pi_{n,m-1} \subset \Pi_{n,m}$  and  $|T_{n,m-1}(\xi_i^{(n,m)})| \leq 1$ , Theorem 1.3 implies that  $|T_{n,m-1}(x)| \leq |T_{n,m}(x)| < 1$  for  $-1 < x < \xi_0^{(n,m)}$ . Hence  $\xi_0^{(n,m-1)} \geq \xi_0^{(n,m)}$ . If they are equal, let  $g(x) = T_{n,m}(x) + T_{n,m-1}(x)$ . Then  $g(\xi_k^{(n,m)})(-1)^{m-k} \geq 0$ ,  $k=0, 1, \dots, m$ , and  $g(\xi_0^{(n,m)}) = g'(\xi_0^{(n,m)}) = 0$ . By counting double zeros as two zeros, one sees that  $g(x)$  has at least  $m+1$  zeros in  $[\xi_0^{(n,m)}, 1]$  which is impossible. Thus we have  $\xi_0^{(n,m)} < \xi_0^{(n,m-1)}$  as desired.

Now suppose for some  $k$ , there is more than one extreme point of  $T_{n,m-1}$  in the interval  $[\xi_k^{(n,m)}, \xi_{k+1}^{(n,m)}]$ . Suppose  $\xi_k^{(n,m)} \leq \xi_j^{(n,m-1)} < \xi_{j+1}^{(n,m-1)} \leq \xi_{k+1}^{(n,m)}$  for some  $j$  and define

$$g(x) = \operatorname{sgn}(T_{n,m}(\xi_{k+1}^{(n,m)})) T_{n,m}(x) + \operatorname{sgn}(T_{n,m-1}(\xi_j^{(n,m-1)})) T_{n,m-1}(x).$$

Then we must have that

$$g(\xi_k^{(n,m)}) \leq 0, \quad g(\xi_j^{(n,m-1)}) \geq 0, \quad g(\xi_{j+1}^{(n,m-1)}) \leq 0, \quad g(\xi_{k+1}^{(n,m)}) \geq 0$$

and hence  $g(x)$  has at least two zeros in  $[\xi_k^{(n,m)}, \xi_{k+1}^{(n,m)}]$  without counting double zeros at the endpoints. Now if  $g(\xi_k^{(n,m)}) = 0$  then  $g(x)$  has a double zero at  $\xi_k^{(n,m)}$  and similarly for  $\xi_{k+1}^{(n,m)}$  if  $k+1 \neq m$ . Since  $g(\xi_i^{(n,m)})(-1)^{m-i} \geq 0$ ,  $i=0, 1, \dots, m$ ,  $g(x)$  has at least  $k$  zeros in  $[\xi_0^{(n,m)}, \xi_k^{(n,m)}]$  and  $m-k-1$  zeros in  $[\xi_{k+1}^{(n,m)}, 1]$ , without counting double zeros at the end points. Hence  $g(x)$  has at least  $m+1$  zeros in  $[\xi_0^{(n,m)}, 1]$  which contradicts the fact that  $0 \neq g(x) \in \Pi_{n,m}$ . This shows that each subinterval  $[\xi_k^{(n,m)}, \xi_{k+1}^{(n,m)}]$  contains at most one  $\xi_j^{(n,m-1)}$ . But there are precisely  $m$  extreme points of  $T_{n,m-1}$  in  $(\xi_0^{(n,m)}, 1]$ . Since  $\xi_{m-1}^{(n,m-1)} = \xi_m^{(n,m)} = 1$  and  $\xi_0^{(n,m)} < \xi_0^{(n,m-1)}$ ,  $(\xi_{k-1}^{(n,m)}, \xi_k^{(n,m)})$  containing no  $\xi_j^{(n,m-1)}$  for some  $0 < k < m$  would imply  $[\xi_{l-1}^{(n,m)}, \xi_l^{(n,m)}]$  containing more than one  $\xi_j^{(n,m-1)}$  for some  $0 < l < m$ . Hence (2.1) must hold. Similar arguments will give (2.2). ■

**THEOREM 2.2.** Let  $T_{n,m}$ ,  $\{\xi_k^{(n,m)}\}_{k=0}^m$ , and  $\{\sigma_k^{(n,m)}\}_{k=1}^m$  be defined as above. Then

$$\frac{(1 - \xi_0^{(n,m)})}{2} \left( \cos \left( \frac{m-k}{m} \pi \right) + 1 \right) + \xi_0^{(n,m)} \leq \xi_k^{(n,m)} \leq \cos \left( \frac{m-k}{m+n} \pi \right) \quad (2.3)$$

and

$$\frac{(1 - \xi_0^{(n,m)})}{2} \left( \cos \left( \frac{m-k+1/2}{m} \pi \right) + 1 \right) + \xi_0^{(n,m)} < \sigma_k^{(n,m)} < \cos \left( \frac{m-k+1/2}{m+n} \pi \right), \quad (2.4)$$

where strict inequalities hold in (2.3) except for  $k=0$ ,  $m$  on the left side and  $k=m$  on the right side.

*Proof.* We omit the superscripts in the following proofs for simplicity. Also, we only show the left side inequality of (2.3) as the proofs of the other inequalities are similar.

Define  $H(x) = ((x+1)/2)^n T_m(2(x-\xi_0)/(1-\xi_0)-1)$ , where  $T_m(x) = \cos(m \cos^{-1}(x))$ , the Chebyshev polynomial of degree  $m$ . Set  $x_k = (1/2)(1-\xi_0)(\cos((m-k)\pi/m) + 1) + \xi_0$ ,  $k=0, 1, \dots, m$ . Then  $\{x_k\}$  consists

of the extreme points of  $T_m(2(x - \xi_0)/(1 - \xi_0) - 1)$ . We first claim  $x_k \leq \xi_k$  for all  $k$ . Suppose this does not hold for some  $k$ , i.e.,  $\xi_k < x_k$ . Then set  $G(x) = (4/(\xi_k + x_k + 2))^n H(x) - T_{n,m}(x) \in \Pi_{n,m}$ . Now, we have

$$\left| \left( \frac{4}{x_k + \xi_k + 2} \right)^n H(x) \right| < 1, \quad \text{for } x \in [\xi_0, \xi_k]$$

and

$$\left| \left( \frac{4}{x_k + \xi_k + 2} \right)^n H(x_j) \right| > 1, \quad \text{for } j \geq k$$

implying that  $G(x)$  has at least  $k$  zeros in  $[\xi_0, \xi_k]$  and  $m - k$  zeros in  $[x_k, 1]$ . Since  $\text{sgn } G(x_k) = -\text{sgn } G(\xi_k)$ ,  $G(x)$  has one more zero in  $(\xi_k, x_k)$ . This shows  $G(x)$  must have at least  $m + 1$  zeros in  $[\xi_0, 1]$  which contradicts the fact that  $G(x) \in \Pi_{n,m}$  does not vanish identically. Thus  $\xi_k \geq x_k$  for all  $k$ .

Now, if  $\xi_k = x_k$  for some  $k \neq 0, m$ , then  $G(x)$  has  $k - 1$  zeros in  $(\xi_0, \xi_{k-1})$ ,  $m - k - 1$  zeros in  $(x_{k+1}, 1)$  and one zero at  $x_k = \xi_k$ . Since  $|(2/(x_k + 1))^n H(x)| < 1$  for  $x < x_k = \xi_k$ ,  $|(2/(x_k + 1))^n H(x_k)| = 1$ , and  $G'(x_k) = (2/(x_k + 1))^n H'(x_k) \neq 0$ , there exists  $\varepsilon > 0$  such that for  $x \in (x_k, x_k + \varepsilon)$ ,  $|(2/(x_k + 1))^n H(x)| > 1$  and  $\text{sgn } H(x) = \text{sgn } H(x_k)$ . Choose any  $x^* \in (x_k, x_k + \varepsilon)$  then we have that  $\text{sgn } G(x^*) = \text{sgn } H(x_k) = -\text{sgn } H(x_{k+1}) = -\text{sgn } G(x_{k+1})$ . That is,  $G(x)$  has one more zero in  $(x^*, x_{k+1})$ . But

$$\begin{aligned} \text{sgn } G(x_{k+1}) &= \text{sgn } H(x_{k+1}) = (-1)^{m-k-1} = \text{sgn } T_{n,m}(\xi_{k-1}) \\ &= -\text{sgn } G(\xi_{k-1}) \end{aligned}$$

implies  $G(x)$  has an odd number of zeros in  $(\xi_{k-1}, x_{k+1})$ . Therefore  $G(x)$  has at least 3 zeros in  $(\xi_{k-1}, x_{k+1})$  and hence at least  $m + 1$  zeros in  $(\xi_0, 1)$ . This gives us our desired contradiction. Hence the inequality is proved. ■

Now we turn to a Kadec-type result for approximation by incomplete polynomials. We first prove a lemma to deal with the technical parts of the theorem.

**LEMMA 2.3.** *Let  $Q_m \in \Pi_{n,m}$  with  $\|Q_m\| < L$  and suppose there exist  $-1 \leq t_0 < \dots < t_m \leq 1$  satisfying  $\text{sgn } Q_m(t_i) = (-1)^i \text{sgn } Q_m(t_0)$  and  $|Q_m(t_i)| = 1$  for  $i = 0, 1, \dots, m$ . Set*

$$A_m = \max_{0 \leq k \leq m} \{ |\cos^{-1}(t_k) - \cos^{-1}(\xi_k^{(n,m)})| \}.$$

Then

$$\Delta_m \leq 3 \left( \frac{\cosh^{-1} L}{m} \right)^{2/3} + o \left( \frac{\cosh^{-1} L + n}{m} \right)^{2/3} \quad (2.5)$$

for  $m$  sufficiently larger than  $\cosh^{-1} L$  (e.g.,  $m \geq (\cosh^{-1} L + 2)^6$ ).

*Proof.* Let  $0 < l < m$  be an integer and define

$$H(x) = T_l \left( 2 \left( \frac{1-x}{1-t_k} \right) - 1 \right) T_{n,m-l} \in \Pi_{n,m}$$

and

$$G(x) = \frac{H(x)}{1+\varepsilon} \pm Q_m(x),$$

where the  $+$  or  $-$  sign preceding  $Q_m(x)$  will be chosen later. Then  $|H(x)| \leq 1$  for  $t_k \leq x \leq 1$ . Noting the fact that  $T_l(y) = \cosh(lt)$  with  $y = \cosh(t)$  for  $y \geq 1$ , we have

$$\frac{T_l(2((1-x)/(1-t_k)) - 1)}{1+\varepsilon} > L$$

$$\text{for } x \leq t_k - \frac{1-t_k}{2} \left[ \cosh \frac{\cosh^{-1} L(1+\varepsilon)}{l} - 1 \right] \triangleq t_k - A_k(\varepsilon).$$

Now we establish that  $t_k \leq \xi_k^{(n,m-l)} + A_k(0)$ ,  $k = 0, 1, \dots, m-l$ , by showing that  $t_k \leq \xi_k^{(n,m-l)} + A_k(\varepsilon)$  for any  $\varepsilon > 0$ .

Suppose this is not true, that is, suppose  $t_k > \xi_k^{(n,m-l)} + A_k(\varepsilon)$  for some  $\varepsilon > 0$  and some  $k$ . Then  $G(x)$  has at least  $m-k$  zeros in  $(t_k, 1]$  and  $k$  zeros in  $(0, \xi_k^{(n,m-l)})$  since  $|Q_m(t_j)| = 1$  with alternating signs,  $|H(x)/(1+\varepsilon)| < 1$  in  $(t_k, 1]$ , and  $|H(\xi_j^{(n,m-l)})| \geq L$  for  $j \geq k$  with alternating signs. Now we can choose a proper sign in the definition of  $G(x)$  to obtain one more zero in  $(\xi_k^{(n,m-l)}, t_k)$ . Doing this gives our desired contradiction. Thus we have proved

$$t_k \leq \xi_k^{(n,m-l)} + A_k(0), \quad 0 \leq k \leq m-l;$$

that is,

$$t_k \leq \xi_k^{(n,m-l)} + \frac{(1-t_k)}{2} \left( \cosh \frac{\cosh^{-1} L}{l} - 1 \right)$$

$$\triangleq \xi_k^{(n,m-l)} + (1-t_k)A, \quad 0 \leq k \leq m-l. \quad (2.6)$$

Next we need to show that

$$\cos^{-1}(t_k) \geq \cos^{-1}(\xi_k^{(n,m)}) - 3 \left( \frac{\cosh^{-1} L}{m} \right)^{2/3} - o \left( \frac{\cosh^{-1} L + n}{m} \right)^{2/3}. \quad (2.7)$$

It is easy to see that we need only establish (2.7) under the assumption  $t_k > \xi_k^{(n,m-l)}$  and  $\xi_k^{(n,m-l)} + (1-t_k)A < 1$ . Let

$$t_k = \xi_k^{(n,m-l)} + (1-t_k)\theta A, \quad (2.8)$$

where  $0 < \theta \leq 1$ . Then

$$\begin{aligned} \cos^{-1}(t_k) &= \cos^{-1}(\xi_k^{(n,m)}) + [\cos^{-1}(\xi_k^{(n,m-l)}) - \cos^{-1}(\xi_k^{(n,m)})] \\ &\quad + [\cos^{-1}(\xi_k^{(n,m-l)} + (1-t_k)\theta A) - \cos^{-1}(\xi_k^{(n,m-l)})]. \end{aligned}$$

But from (2.3)

$$|\cos^{-1}(\xi_k^{(n,m-l)}) - \cos^{-1}(\xi_k^{(n,m)})| \leq \frac{mn + kl - kn}{m(m+n-l)} \pi$$

and by the Mean-Value Theorem, (2.3), (2.8), and for  $A < 1$  we have

$$\begin{aligned} &|\cos^{-1}(\xi_k^{(n,m-l)}) - \cos^{-1}(\xi_k^{(n,m-l)} + (1-t_k)\theta A)| \\ &= \left| \frac{(1-t_k)\theta A}{\sqrt{1 - (\xi_k^{(n,m-l)} + (1-t_k)\theta A)^2}} \right| \\ &= \frac{\sqrt{1-t_k}\theta A}{\sqrt{1 + \xi_k^{(n,m-l)} + (1-t_k)\theta \mu A + (1+t_k)(1-\mu)\theta A - (1-t_k)(1-\mu)^2\theta^2 A^2}} \\ &\leq \frac{A}{\sqrt{1 + \xi_k^{(n,m-l)}}} \leq \frac{A}{\sin(k\pi/2(m-l))} \leq \frac{A}{(k/(m-l))}, \end{aligned} \quad (2.9)$$

where  $0 < \mu < 1$ . Set  $l = [\frac{1}{2}m^{5/6}(\cosh^{-1} L)^{2/3}/(k+1)^{1/2}]$  and assume  $(\cosh^{-1} L)^6 \leq m$ . Then for  $1 \leq k \leq m-l$  we have that

$$\begin{aligned} \frac{mn + kl - kn}{m(m+n-l)} \pi &\leq \frac{n}{m+n-l} \pi + \frac{k^{1/2}(\cosh^{-1} L)^{2/3}}{2m^{1/6}(m+n-l)} \pi \\ &\leq o \left( \frac{n}{m+n} \right)^{2/3} + \frac{\pi}{2} \left( \frac{\cosh^{-1} L}{m} \right)^{2/3} + o \left( \frac{\cosh^{-1} L}{m} \right)^{2/3} \end{aligned}$$

and

$$\begin{aligned} \frac{m-l}{k} A &\leq \frac{m-l}{2k} \left[ \frac{1}{2} \left( \frac{\cosh^{-1} L}{l} \right)^2 + \sum_{j=2}^{\infty} \frac{1}{(2j)!} \left( \frac{\cosh^{-1} L}{l} \right)^{2j} \right] \\ &\leq \left( \frac{\cosh^{-1} L}{m} \right)^{2/3} + o \left( \frac{\cosh^{-1} L}{m} \right). \end{aligned}$$

Hence,

$$\begin{aligned} \cos^{-1}(t_k) &\geq \cos^{-1}(\xi_k^{(n,m)}) - \frac{mn + kl - kn}{m(m+n-l)} \pi - \frac{m-l}{k} A \\ &\geq \cos^{-1}(\xi_k^{(n,m)}) - 3 \left( \frac{\cosh^{-1} L}{m} \right)^{2/3} - o \left( \frac{\cosh^{-1} L + n}{m} \right)^{2/3}. \quad (2.10) \end{aligned}$$

For  $k \geq m-l$  we have that

$$\begin{aligned} \cos^{-1}(t_k) &\geq 0 \geq \cos^{-1}(\xi_k^{(n,m)}) - \frac{m-k}{m} \pi \\ &\geq \cos^{-1}(\xi_k^{(n,m)}) - \frac{l}{m} \pi \geq \cos^{-1}(\xi_k^{(n,m)}) - \frac{(\cosh^{-1} L)^{2/3}}{2m^{1/6}(m-l)^{1/2}} \pi \\ &= \cos^{-1}(\xi_k^{(n,m)}) - \frac{\pi}{2} \left( \frac{\cosh^{-1} L}{m} \right)^{2/3} - o \left( \frac{\cosh^{-1} L}{m} \right)^{2/3}. \end{aligned}$$

For  $k=0$ , from Theorem 2.1,  $\xi_0^{(n,m-l)} \geq 2(n/(n+m-l))^2 - 1$  and hence in view of (2.9)

$$\begin{aligned} \cos^{-1}(\xi_0^{(n,m-l)}) &= \cos^{-1}(\xi_0^{(n,m-l)} + (1-t_k)\theta A) \\ &\leq \frac{A(n+m-l)}{n} \quad \text{for } n \geq 0. \end{aligned}$$

When  $n=0$ ,  $\xi_0^{(n,m)} = -1$  and  $\cos^{-1}(-1) = \cos^{-1}(-1 + (1-t_k)\theta A) \leq 2\sqrt{A}$  for sufficiently small  $A$ , and  $\sqrt{A} = o(1/m)^{2/3}$ . This makes (2.10) valid for  $k=0$  (replace  $(m-l)/k$  by  $(n+m-l)A/n$  or  $2\sqrt{A}$ ) and hence (2.7) is established for  $0 \leq k \leq m$ .

Similarly we can show

$$\cos^{-1}(t_k) \leq \cos^{-1}(\xi_k^{(n,m)}) + 3 \left( \frac{\cosh^{-1} L}{m} \right)^{2/3} + o \left( \frac{\cosh^{-1} L + n}{m} \right)^{2/3}. \quad (2.11)$$

Combining (2.7) and (2.11), we obtain (2.5). ■

We now state our main theorem.

**THEOREM 2.4.** *Let  $f \in C[-1, 1]$ ,  $f(-1) = 0$  if  $n \neq 0$ , and  $f \notin \bigcup_{m=1}^{\infty} \Pi_{n,m}$ . Suppose  $P_m \in \Pi_{n,m}$  is the best approximation to  $f$  from  $\Pi_{n,m}$ . Set  $E_m = \|f - P_m\|$  and let  $-1 \leq x_0 < x_1 < \cdots < x_{m+1} \leq 1$  be the Chebyshev alternation points of  $f - P_m$ , that is,*

$$P_m(x_i) - f(x_i) = \sigma(-1)^i E_m, \quad \sigma = \pm 1.$$

Set

$$A_m = \max_{0 \leq k \leq m+1} \{ |\cos^{-1}(x_k) - \cos^{-1}(\xi_k^{(n,m+1)})| \}.$$

Then

$$\lim_{m \rightarrow \infty} \frac{A_m m^{2/3}}{(\log m)^{2/3}} \leq 3. \quad (2.12)$$

*Proof.* Let  $Q_{m+1} = (P_{m+1} - P_m)/(E_m - E_{m+1}) \in \Pi_{n,m+1}$ . By the assumption that  $f \notin \bigcup_{m=1}^{\infty} \Pi_{n,m}$ ,  $Q_{m+1}$  is well defined for infinite many  $m$ ,

$$\|Q_{m+1}(x)\| \leq \frac{E_m^{m+1} + E_m}{E_m - E_{m+1}} \triangleq L_m$$

and

$$|Q_{m+1}(x_k)| \geq 1$$

whenever  $E_{m+1} < E_m$ .

Let  $t_k$ ,  $k = 0, 1, \dots, 2m+3$ , be the roots of the equation of  $|Q_{m+1}(x)| = 1$  with  $t_{2k} \leq x_k \leq t_{2k+1}$ . Note that  $Q_{m+1}(t_{2k}) = Q_{m+1}(t_{2k+1}) = -Q_{m+1}(t_{2k+2}) = -Q_{m+1}(t_{2k+3})$ ,  $k = 0, \dots, m$ , and then by Lemma 2.3 we have that

$$A_m \leq 3 \left( \frac{\cosh L_m}{m+1} \right)^{2/3} + o \left( \frac{\cosh^{-1} L_m + n}{m+1} \right)^{2/3}.$$

We claim that  $\lim_{m \rightarrow \infty} (\log L_m)/\log m^\alpha \leq 1$ , for any  $\alpha > 1$ . Supposing this is not true, there exist  $m_0$  and  $\alpha > 1$  such that  $L_m > m^\alpha$  for all  $m \geq m_0$  and hence  $E_{m+1} > [(m^\alpha - 1)/(m^\alpha + 1)] E_m$ . Hence

$$\overline{\lim}_{m \rightarrow \infty} \frac{E_m}{E_{m_0}} \geq \overline{\lim}_{m \rightarrow \infty} \prod_{k=m_0}^{m-1} \left( \frac{k^\alpha - 1}{k^\alpha + 1} \right) > 0$$

contradicting the fact that  $\lim_{m \rightarrow \infty} E_m = 0$ . Thus there exists a subsequence  $\{m_k\}$  such that

$$\lim_{k \rightarrow \infty} \frac{\log L_{m_k}}{\log m_k} = \lambda \leq 1.$$



Note that this guarantees that  $(\cosh^{-1} L_{m_k} + 2)^6 \leq m_k$  holds for large  $k$ , which shows that the hypothesis of Lemma 2.3 is satisfied. Thus we have

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{\Delta_m m^{2/3}}{(\log m)^{2/3}} &\leq \lim_{k \rightarrow \infty} \frac{\Delta_{m_k} m_k^{2/3}}{(\log m_k)^{2/3}} \\ &\leq \lim_{k \rightarrow \infty} \frac{3(\cosh^{-1} L_{m_k}/(m_k + 1))^{2/3} m_k^{2/3}}{(\log m_k)^{2/3}} = 3\lambda^{2/3} \leq 3. \quad \blacksquare \end{aligned}$$

**COROLLARY 2.5.** *Let  $f \in C[0, \pi]$  and  $Q_m$  be the best approximation to  $f$  from the set of all even trigonometric polynomials of degree  $\leq m$ . Let*

$$t_0^{(m)} < t_1^{(m)} < \dots < t_{m+1}^{(m)}$$

*be a set of alternating extreme points of the error curve and set*

$$\delta_m = \max_{0 \leq k \leq m+1} \left\{ \left| t_k^{(m)} - \frac{k\pi}{m+1} \right| \right\}.$$

*Then*

$$\lim_{m \rightarrow \infty} \frac{\delta_m m^{2/3}}{(\log m)^{2/3}} \leq 3. \quad (2.13)$$

This corollary improves Kadec's result in [3] where he proved

$$\lim_{m \rightarrow \infty} \delta_m m^{1/2-\varepsilon} = 0, \quad \text{for any } \varepsilon > 0.$$

If  $f$  is analytic on  $[-1, 1]$  the rate in (2.12) can be slightly improved.

**THEOREM 2.6.** *Let the hypothesis of Theorem 2.4 hold and assume, in addition, that  $f$  is analytic on  $[-1, 1]$  and  $f^{(k)}(-1) = 0$  for  $k = 0, 1, \dots, n-1$ . Then there exists a constant  $c$  such that*

$$\lim_{m \rightarrow \infty} \Delta_m m^{2/3} \leq c. \quad (2.14)$$

*Proof.* Let  $L_m$  and  $E_m$  be defined as before. It can be easily shown (see [11]) that if a function  $f$  is analytic on  $[-1, 1]$  and  $f^{(k)}(-1) = 0$ ,  $k = 0, 1, \dots, n-1$  then  $\lim_{m \rightarrow \infty} m^{n/2} \sqrt{E_m} = q < 1$ . This implies that

$$\lim_{m \rightarrow \infty} \frac{E_{m+1}}{E_m} \leq q$$

because otherwise  $E_{m+1} \geq q'E_m$  for all  $m \geq m_0$  and some  $q' > q$  implies  $E_m \geq (q')^{m-m_0} E_{m_0}$  which contradicts  $\lim_{m \rightarrow \infty} \sqrt[m+n]{E_m} = q < q'$ . Thus we have

$$\lim_{m \rightarrow \infty} L_m \leq \lim_{m \rightarrow \infty} \frac{1 + E_{m+1}/E_m}{1 - E_{m+1}/E_m} \leq \frac{2}{1-q}.$$

In view of the proof of Theorem 2.4 and the above inequality, (2.14) follows. ■

In [10],  $\tilde{\delta}_m = \max_{0 \leq k \leq m+1} \{|t_k^{(m)} - t_{k+1}^{(m)}|\}$  was considered and it was proved that

$$\lim_{m \rightarrow \infty} \tilde{\delta} \frac{m}{\log m} < \infty. \quad (2.15)$$

Similar results can be shown for best approximation by incomplete polynomials. Recently it was shown in [2] that

$$\lim_{m \rightarrow \infty} \delta_m \frac{m}{\log m} < \infty \quad (2.16)$$

under the stronger hypothesis that  $f$  has an analytic continuation in the closed ellipse with foci  $\pm 1$  and sum of semiaxes  $r > 1/\rho$ , where  $\rho \approx 0.12366\dots$ . Note, it is an open question whether these rates are sharp or not. Evidently for some functions, for example,  $f(x) = \sum_{n=1}^{\infty} (1/3)^n \cos(3^n \cos^{-1} x)$ ,  $\delta_m = O(1/m)$ . However, a counterexample given by G. G. Lorentz in [6] showed that  $\liminf$  cannot be replaced by limit even if we only consider the set of all entire functions.

#### ACKNOWLEDGMENT

The material in this paper is primarily extracted from my dissertation. I thank my thesis advisor, Professor G. D. Taylor for his unlimited help throughout my graduate years at Colorado State University.

#### REFERENCES

1. E. W. CHENEY, "Introduction to Approximation Theory," McGraw-Hill, New York, 1966.
2. R. GROTHMANN, On the real CF-Method for polynomial approximation and strong unicity constants, *J. Approx. Theory* **55** (1988), 86–103.

3. M. I. KADEC, On the distribution of points of maximal deviation in the approximation of continuous functions by polynomials, *Uspekhi Mat. Nauk* **15** (1960), 199–202.
4. G. G. LORENTZ, Approximation by incomplete polynomials (problem and results), in "Pade and Rational Approximation: Theory and Application" (E. B. Saff and R. S. Varga, Eds.), pp. 289–302, Proceedings of an International Symposium, Tampa, 1976, Academic Press, London/New York, 1977.
5. G. G. LORENTZ, Problems for incomplete polynomials, in "Approximation Theory, IV," pp. 41–73, Academic House, New York, 1983.
6. G. G. LORENTZ, Distribution of alternation points in uniform polynomials approximation, *Proc. Amer. Math. Soc.* **92** (1984), 257–263.
7. E. B. SAFF AND R. S. VARGA, The sharpness of Lorentz's theorem on incomplete polynomials, *Trans. Amer. Math. Soc.* **249** (1979), 163–186.
8. E. B. SAFF AND R. S. VARGA, On incomplete polynomials, *Numer. Meth. Approx. Theor.* **42**, No. 4 (1978), 281–298.
9. E. B. SAFF AND R. S. VARGA, On incomplete polynomials, II. *Pacific J. Math.* **92** (1981), 161–172.
10. S. TASHEV, On the distribution of the points of maximal deviation for the polynomials of best Chebyshev and Hausdorff approximation, in "Approx. and Function Space, Proc. Conf., Gdansk, 1979" (Z. Ciesielski, Ed.), pp. 791–799, 1981.
11. C. YANG, "Contributions to the Theory of Approximation by Polynomials," Doctoral Dissertation in Mathematics, Colorado State University, 1989.